Matrices (2×2)

Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be a vector and let

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix.

Then we can view A as a function from 2D vectors to 2D vectors (i.e., $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$).

A sends $\vec{v}$ to the vector $A\vec{v}$:

$$A\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Ex: Compute $A\vec{v}$ for the given A & $\vec{v}$

ⓐ $A = \begin{pmatrix} -4 & 2 \\ 1 & 6 \end{pmatrix}$ ⓑ $A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Sol:

ⓐ $$A\vec{v} = \begin{pmatrix} -4 & 2 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -12-4 \\ 3-12 \end{pmatrix} = \begin{pmatrix} -16 \\ -9 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 3+4 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

Dot Product

We won't use the dot product for anything but notational simplicity.

2D-vectors $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\vec{v} \cdot \vec{w} = ax + by$$

Matrix Multiplication

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}$ where

$\vec{r}_1 = (a_{11} \ a_{12})$ & $\vec{r}_2 = (a_{21} \ a_{22})$ are the rows of A.

Likewise, let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 \end{pmatrix}$

where $\vec{c}_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$ & $\vec{c}_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$ are the columns of B.

The product AB is defined by

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \cdot \vec{c}_1 & \vec{r}_1 \cdot \vec{c}_2 \\ \vec{r}_2 \cdot \vec{c}_1 & \vec{r}_2 \cdot \vec{c}_2 \end{pmatrix}$$

$$= \begin{pmatrix} A\vec{c}_1 & A\vec{c}_2 \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Ex: Compute AB & BA for the matrices

$$A = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$$

Sol: $$AB = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 10+4 & 0+3 \\ 6-8 & 0-6 \end{pmatrix} = \begin{pmatrix} 14 & 3 \\ -2 & -6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 10+0 & 2+0 \\ 20+9 & 4-6 \end{pmatrix} = \begin{pmatrix} 12 & 2 \\ 29 & -2 \end{pmatrix}$$

Notice that $AB \neq BA$.

Inverse of a Matrix

If A is a matrix, the inverse of A is a matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$, where I is the identity matrix, $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and if $ad - bc \neq 0$,

then A^{-1} exists and is given by:

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Ex: Compute the inverse of the given matrices

ⓐ $A = \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}$ ⓑ $B = \begin{pmatrix} 2 & -4 \\ 4 & -6 \end{pmatrix}$

Sol:

ⓐ $(8)(4) - (6)(5) = 32 - 30 = 2$

$$A^{-1} = \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -6 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -5/2 & 4 \end{pmatrix}$$

ⓑ $(2)(-6) - (-4)(4) = -12 + 16 = 4$

$$B^{-1} = \begin{pmatrix} 2 & -4 \\ 4 & -6 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} -6 & 4 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -3/2 & 1 \\ -1 & 1/2 \end{pmatrix}$$

<u>Ex</u>: The product of the matrices in the previous example is

$$AB = \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 4 & -6 \end{pmatrix} = \begin{pmatrix} 40 & -68 \\ 26 & -44 \end{pmatrix}$$

and so the inverse of the product is

$$(AB)^{-1} = \frac{1}{-1760+1768}\begin{pmatrix} -44 & 68 \\ -26 & 40 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -44 & 68 \\ -26 & 40 \end{pmatrix} = \begin{pmatrix} -11/2 & 17/2 \\ -13/4 & 5 \end{pmatrix}$$

Which of the following is true?

ⓐ $(AB)^{-1} = A^{-1}B^{-1}$ ⓑ $(AB)^{-1} = B^{-1}A^{-1}$ ⓒ Neither

<u>Sol</u>: The answer is ⓑ

$$B^{-1}A^{-1} = \begin{pmatrix} -3/2 & 1 \\ -1 & 1/2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -5/2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3-\frac{5}{2} & \frac{9}{2}+4 \\ -2-\frac{5}{4} & 3+2 \end{pmatrix} = \begin{pmatrix} -11/2 & 17/2 \\ -13/4 & 5 \end{pmatrix} = (AB)^{-1}$$

<u>Fact</u>: $A\vec{0} = A\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$

So A <u>always</u> fixes the origin!

How does $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transform the plane?

Notice that any vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ can be written in terms of $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as

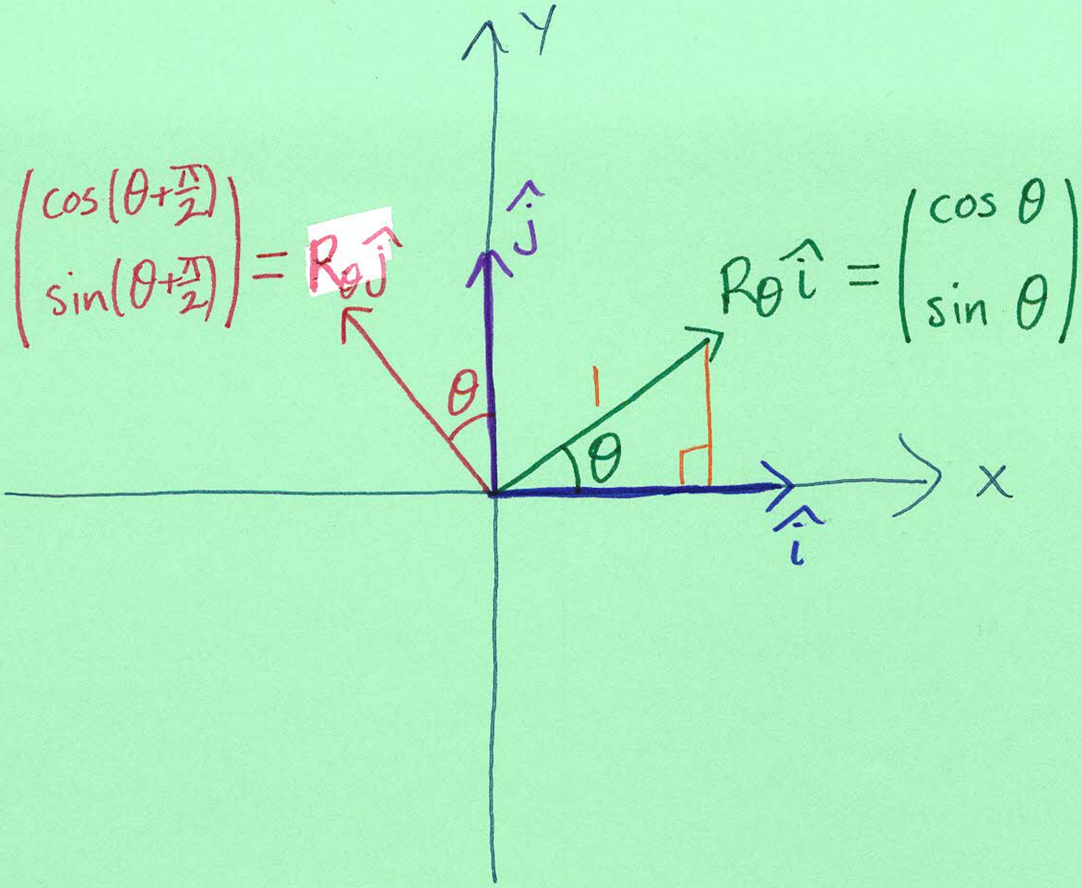
$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = x\hat{i} + y\hat{j}$, so it suffices to understand $A\hat{i}$ & $A\hat{j}$ since $A\vec{v} = xA\hat{i} + yA\hat{j}$.

$$A\hat{i} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$A\hat{j} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

So, $A = \begin{pmatrix} A\hat{i} & A\hat{j} \end{pmatrix}$.

Rotations: Suppose we want to rotate the plane counterclockwise by θ (rotation about the origin). By the above, we need to check what happens to $\hat{i}$ & $\hat{j}$:



Using $\sin(\theta+\frac{\pi}{2}) = \cos\theta$ & $\cos(\theta+\frac{\pi}{2}) = -\sin\theta$, we find that $R_\theta \hat{j} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$, so the rotation matrix R_θ is given by

$$\boxed{R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$